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Density minoration of a strongly non-degenerated random variable

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Abstract

We obtain a lower bound for the density of a d-dimensional random variable on the Wiener space under exponential moment condition of the divergence of covering vector fields.

Keywords: Malliavin calculus; Lower bounds for densities; Divergence; Riesz transforms

1. Introduction

Finding lower bounds for densities of random variables on the Wiener space has been a current subject of research in probability theory for the last twenty years. As is well known, the use of stochastic calculus of variations (Malliavin calculus) is a tool for studying existence, smoothness of densities of random variables on the Wiener space, as well as finding explicit lower bounds. The work by Kusuoka and Stroock became the starting point of the use of Malliavin calculus to obtain lower bounds for densities. In [9], they obtained a lower bound of Gaussian type for the density of a uniformly hypoelliptic diffusion whose drift is a smooth combination of its diffusion coefficient. Their results are the first known extensions of the analytical results obtained in [5,15]. Fifteen years later, Kohatsu-Higa [7,8] extended the result by Kusuoka and Stroock by giving

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a general definition of uniformly elliptic random variables on the Wiener space and obtained Gaussian type lower bounds for those random variables. He applied his results to the solution of the non-linear heat equation and to non-homogeneous uniformly elliptic diffusions. The result by Kohatsu-Higa was applied in [4] to solutions of non-linear hyperbolic SPDEs to derive results on potential theory. Later on, Bally [2] relaxed the uniformly elliptic condition of Kohatsu-Higa. The ideas of Bally were used in [6] to obtain a lower bound for the density of a non-linear Landau process with a degenerate diffusion coefficient.

In this paper we are interested in finding lower bounds for densities of abstract random variables on the Wiener space; we extend the one-dimensional result obtained in [13], which is a lower bound for the density of a real random variable F on the Wiener space, under an exponential moment on the divergence of a *covering vector field* of F; the methodology used in [13] was the resolution of a one-dimensional variational problem. The present work will use the following tricks to solve the case of R^d valued random variables: radial averaging combined with Riesz transform estimates on the sphere and Riesz transform estimates on balls.

2. Notations and main theorem

We first introduce some elements of the differential calculus on Gaussian probability spaces (see for instance [11,14]). Let $W = \{W(h),\ h \in H\}$ be an isonormal Gaussian process associated with a Hilbert space H.

Let S denote the class of smooth random variables of the form $F = f(W(h_1), \dots, W(h_n))$, where $h_1, \dots, h_n$ are in H, $n \geqslant 1$, and f belongs to $C_P^\infty(R^n)$, the set of functions f such that f and all its partial derivatives have at most polynomial growth. Given F in S, its *derivative* is the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \partial_i f\big(W(h_1), \dots, W(h_n)\big)h_i.$$

For $h \in H$ fixed, we define the operator D_h on the set S by $D_h F = \langle DF, h\rangle_H$.

More generally, the kth order derivative of $F \in S$ is obtained by iterating the derivative operator k times and is denoted $D^k F$. Then for every $p \geqslant 1$ and any natural number k, we denote by D_k^p the clousure of S with respect to the norm $\|\cdot\|_{k,p}$ defined by

$$\|F\|_{k,p}^p = E\big[|F|^p\big] + \sum_{j=1}^{k} E\big[\big\|D^j F\big\|_{H^{\otimes j}}^p\big].$$

The derivative operator D is a closed and unbounded operator with values in $L^2(\Omega; H)$; it is defined on the dense subset D_1^2 of $L^2(\Omega)$. We denote by δ the adjoint of the operator D, which is an unbounded operator on $L^2(\Omega, H)$ taking values in $L^2(\Omega)$. In particular, if u belongs to $\operatorname{Dom}\delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the duality relation:

$$E\big[F\delta(u)\big] = E\big[\langle DF, u\rangle_H\big], \quad \text{for any } F \in D_1^2. \tag*{(2.1)$_a$}$$

The operator δ is called the *divergence* operator.

An important application of Malliavin calculus is the following criterion for existence and smoothness of densities:

We say that an R^d-valued random variable F is *non-degenerated* if $F \in D_2^\infty$ and $(\det \gamma_F)^{-1} \in L^p(\Omega)$ for all $p > 1$, where γ_F denotes the Malliavin matrix of F, that is,

$$(\gamma_F)_{ij} = \langle DF_i, DF_j \rangle_H, \quad 1 \leqslant i, j \leqslant d. \tag{2.1$_b$}$$

Then *the law of a non-degenerated R^d-valued random variable F has a probability density function which is Hölderian* (see for instance [11, p. 72, Theorem 4.1], [14]).

Consider a non-degenerated R^d-valued random variable F. Let us recall the definition of system of *covering vector fields of F* (see for instance [12]):

An $H^{\otimes d}$-valued random variable $(A_1, \dots, A_d)$ is a system of covering vector fields of F if $A_i \in \operatorname{Dom} \delta$, and if

$$(\partial_i \phi) \circ F = D_{A_i}(\phi \circ F) \tag{2.1$_c$}$$

for all ϕ smooth and $i = 1, \dots, d$.

For instance, denoting $\gamma_F^{i,j}$ the inverse of the Malliavin matrix γ_F, then

$$A_i := \sum_j \gamma_F^{i,j} DF_j \tag{2.1$_d$}$$

is system of covering vector fields of F (there exist many other possible choices of covering vector fields of F).

Then, for any system of covering vector fields, we have (see for instance [11,12,14])

$$E\big[\partial_i \phi(F)\big] = E\big[\phi(F)\delta(A_i)\big], \quad \text{for all } i = 1, \dots, d. \tag{2.1$_e$}$$

The main theorem of this paper is the following (see [13] for the one-dimensional case).

Theorem. *Assume that there exists a system of covering vector fields A_i of F, $\gamma > 1$ and $c > 0$ such that,*

$$E\left[\exp\left(c \sum_{i=1}^d |\delta(A_i)|^\gamma\right)\right] < \infty. \tag{2.2$_a$}$$

Then the law of F has a probability density function p in R^d such that there exists $c_\gamma > 0$ satisfying

$$p(x) \geqslant c_\gamma \exp\big(-c_\gamma \|x\|^{\frac{\gamma}{\gamma-1}}\big), \quad \textit{for all } x \in R^d. \tag{2.2$_b$}$$

Reduction to an inequality on $\mathbf{R^d}$

Let $\phi : R^d \mapsto R$ be smooth. By the integration by parts formula and $(2.1)_e$,

$$\int_{R^d} \phi(x) \partial_i p(x)\, dx = \int_{R^d} \phi(x) E\big[\delta(A_i) \bigm| F = x\big] p(x)\, dx.$$

Therefore,

$$\frac{\partial_i p}{p}(x) = E\big[\delta(A_i) \bigm| F = x\big], \quad \text{for all } i = 1, \dots, d. \tag{2.3$_a$}$$

Moreover, writing $q_i := E[\delta(A_i) \mid F]$, for any $\gamma > 1$, it holds that

$$\frac{\|\nabla p\|^\gamma}{p^\gamma}(F) = \|q\|^\gamma \leqslant \sum_{i=1}^d |q_i|^\gamma,$$

and, thus, for any $\gamma > 1$ and $c > 0$, using Jensen's inequality,

$$E\left[\exp\left(c\frac{\|\nabla p\|^\gamma}{p^\gamma}(F)\right)\right] \leqslant E\left[\exp\left(c\sum_{i=1}^d |q_i|^\gamma\right)\right] \leqslant E\left[\exp\left(c\sum_{i=1}^d \big|\delta(A_i)\big|^\gamma\right)\right]. \tag{2.3$_b$}$$

Hence, hypothesis $(2.2)_a$, implies that there exists $\gamma > 1$ and $c > 0$ such that

$$\int\limits_{R^d} \exp\big(c\big\|\nabla \log p(x)\big\|^\gamma\big) p(x)\,dx < +\infty.$$

That is, we have that

$$\int\limits_{R^d} \exp\big(c\big\|\nabla f(x)\big\|^\gamma - f(x)\big)\,dx < +\infty, \quad \text{where } f = -\log p. \tag{2.3$_c$}$$

We want to prove that $(2.3)_c$ implies that

$$f(x) \leqslant c_\gamma \|x\|^{\frac{\gamma}{\gamma-1}}, \quad \text{for all } x \in R^d, \qquad \|x\| \geqslant 1. \tag{2.3$_d$}$$

This would prove $(2.2)_b$.

We have reduced our problem to an implication on R^d: does $(2.3)_c \to (2.3)_d$? The methodology used to prove this implication will depend on three steps: a radial averaging method, an estimation of Riesz transform on the unit sphere of R^d and, finally, an estimation of Riesz transform on the unit ball of R^d.

3. Radial averaging

For any C^1 positive function f on R^d, $\gamma > 1$, and $c > 0$, we define the functional

$$I_\gamma(f) := \int\limits_{R^d} \exp\big(c\big\|\nabla f(x)\big\|^\gamma\big)\exp\big(-f(x)\big)\,dx. \tag{3.1$_a$}$$

The average of a function over the orthogonal group G with respect its Haar measure σ is defined as

$$f^*(x) := \int_G f(gx)\sigma(dg), \quad \text{for all } x \in R^d. \tag{3.1$_b$}$$

Theorem. *For any C^1 positive function f on R^d and for all $\gamma > 1$,*

$$I_\gamma(f^*) \leqslant I_\gamma(f). \tag{3.1$_c$}$$

Proof. In two dimensions, the average of f can be written as

$$f^*(x) := \frac{1}{2\pi}\int_0^{2\pi} f(R_\theta x)\,d\theta, \quad \text{for all } x \in R^2,$$

where R_θ is the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let $\gamma > 1$ be fixed and let f_1, f_2 be C^1 functions on R^d. Then

$$\left\|\nabla\frac{f_1+f_2}{2}\right\|^\gamma \leqslant \frac{1}{2}\|\nabla f_1\|^\gamma + \frac{1}{2}\|\nabla f_2\|^\gamma, \tag{3.2$_a$}$$

as

$$\left\|\nabla\frac{f_1+f_2}{2}\right\| \leqslant \frac{1}{2}\big(\|\nabla f_1\| + \|\nabla f_2\|\big)$$

and the function $\xi \mapsto \xi^\gamma$ is convex. Hence,

$$\begin{aligned} I_\gamma\left(\frac{f_1+f_2}{2}\right) &\leqslant \int_{R^d} \exp\left(\frac{c}{2}\|\nabla f_1\|^\gamma\right)\exp\left(\frac{c}{2}\|\nabla f_2\|^\gamma\right)\exp\left(-\frac{1}{2}f_1\right)\exp\left(-\frac{1}{2}f_2\right)dx \\ &\leqslant \sqrt{I_\gamma(f_1)\times I_\gamma(f_2)}, \end{aligned} \tag{3.2$_b$}$$

the last inequality obtained using Cauchy–Schwarz inequality.

We set $\Phi_x(\theta) = f(R_\theta x)$ and

$$f_n(x) = \frac{1}{2^n}\sum_{0\leqslant k<2^n} \Phi_x\left(\frac{k\pi}{2^n}\right), \quad n \geqslant 0.$$

Iterating the last inequality in R^2 it holds that for all $n \geqslant 0$, $I_\gamma(f_n(x)) \leqslant I_\gamma(f)$. In particular,

$$\liminf_{n\to\infty} I_\gamma\big(f_n(x)\big) \leqslant I_\gamma(f). \tag{3.2$_c$}$$

Moreover, for all $x \in R^2$, $\lim_{n\to+\infty} f_n(x) = f^*(x)$. Hence,

$$\lim_{n\to+\infty} \exp\bigl(c\bigl\|\nabla f_n(x)\bigr\|^\gamma\bigr)\exp\bigl(-f_n(x)\bigr) = \exp\bigl(c\bigl\|(\nabla f)^*(x)\bigr\|^\gamma\bigr)\exp\bigl(-f^*(x)\bigr).$$

By Fatou's lemma,

$$I_\gamma(f^*) \leqslant \int_{R^2} \lim_{n\to+\infty} \exp\bigl(c\bigl\|\nabla f_n(x)\bigr\|^\gamma\bigr)\exp\bigl(-f(x)\bigr)\,dx \leqslant \liminf_{n\to\infty} I_\gamma(f_n). \qquad (3.2)_d$$

This proves the result in dimension 2. In dimension d, we pick an orthonormal basis of the Lie algebra of G and we average on the rotations corresponding to each of these elements. □

Corollary. *Let f be a C^1 positive function on R^d such that for some $\gamma > 1$ and $c > 0$, $I_\gamma(f) < +\infty$. Then, there exists a constant $c_\gamma > 0$ such that for all $x \in R^d$ sufficiently large,*

$$f^*(x) \leqslant c_\gamma \|x\|^{\frac{\gamma}{\gamma-1}}. \qquad (3.2)_e$$

Proof. By the last Theorem, $I_\gamma(f^*) < +\infty$. Because f^* is a radial, changing variables $\|x\| = r$,

$$I_\gamma(f^*) = \omega_d \int_0^{+\infty} \exp\bigl(c\bigl(f'(r)\bigr)^\gamma\bigr)\exp\bigl(-f(r)\bigr)r^{d-1}\,dr.$$

Because the last integral can be bounded below by

$$\int_1^{+\infty} \exp\bigl(c\bigl(f'(r)\bigr)^\gamma\bigr)\exp\bigl(-f(r)\bigr)\,dr,$$

Appendix A implies the desired result. □

4. Riesz transform on the sphere

Set G the orthogonal group of R^d; set $\mathcal{G}$ its Lie algebra of infinitesimal derivations on the left; then $\mathcal{G}$ is isomorphic to $d \times d$ antisymmetric matrices a, $a^T + a = 0$.

The unit sphere S of R^d is an homogeneous space under the action of G. Choosing a point $s_0 \in S$ we get a map $\Psi : G \mapsto S$ defined by $\Psi(g) = g^{-1}(s_0)$; functional spaces on S are lifted by Ψ^* to functional spaces on G. Instead of establishing Riesz transform on S it will be at the same time easier and more general to establish Riesz transform on G.

Denote by ad the adjoint action of $\mathcal{G}$ on itself:

$$\mathrm{ad}(a)(X) = [a, X] = aX - Xa, \quad a, X \in \mathcal{G}.$$

For $i < j$ set $a_{i,j} \in \mathcal{G}$ the matrix having all its coefficients equal to zero at the exception of the i (or j) line and of the j (or i) column, the absolute value of the non-vanishing coefficients being equal to 1. Then $\{a_{i,j}\}_{i<j}$ constitutes a basis of $\mathcal{G}$; we define an euclidean metric on $\mathcal{G}$ by imposing that the basis $\{a_{i,j}\}_{i<j}$ will be orthonormal.

The euclidean structure so choosen on $\mathcal{G}$ defines on G a structure of Riemannian manifold. Using [3] formula $(2.1)_d$ we obtain that the Ricci tensor of G is equal to

$$\mathrm{Ricci} = -\frac{1}{4}\sum_{i<j}\left[\mathrm{ad}(a_{i,j})\right]^2 = \frac{d-1}{4} \times \mathrm{Identity}. \tag{4.1$_a$}$$

Set Δ_0 the Laplace Beltrami operator of G operating on functions; it generates a diffusion process on G; it can be proved (see [3]) that this diffusion can be defined by solving the following Stratonovitch stochastic differential equation

$$dg_b(t) = \left(\sum_{i<j} a_{i,j} \circ db_{i,j}(t)\right) g_b(t), \qquad g_b(0) = \mathrm{Identity}, \tag{4.1$_b$}$$

where $b_{i,j}$ are independent scalar valued Brownian motions. The law of $g_b(t)$ converges when $t \to \infty$ towards the Haar measure σ of G.

Using the right invariant parallelism on G it is possible to construct a 1-differential form on G to a map $G \mapsto \mathcal{G}$.

Set Δ_1 the Hodge–Laplace–Beltrami operator operating on 1-differential forms, set d the exterior differential sending p-differential forms into $p+1$-differential forms and set d^* its adjoint. We then have

$$\Delta_0 = d^* d, \qquad \Delta_1 = dd^* + d^* d,$$

together with the basic Hodge commutations:

$$d\Delta_0 = \Delta_1 d, \qquad \Delta_0 d^* = d^* \Delta_1. \tag{4.1$_c$}$$

Theorem *(Riesz transform). Let f be a C^1 function on G with vanishing mean value. Then there exists a kernel $K \in L^1_\sigma(G;\mathcal{G})$ such that*

$$f(g_0) = \nabla f * K(g_0) := \int_G \nabla f\left(g^{-1}g_0\right) K(g)\, \sigma(dg), \tag{4.2$_a$}$$

where

$$\int_G \left\|K(g)\right\|^p_{\mathcal{G}}\, \sigma(dg) =: \|K\|^p_{L^p_\sigma(G;\mathcal{G})} < \infty,$$

for all $p < \frac{d'+2}{d'+1}$, where $d' = \frac{d(d-1)}{2}$ is the dimension of G.

Proof. Set $\lambda = \nabla f = df$. Then,

$$\Delta_0 f = d^* \lambda.$$

Therefore,

$$f = \Delta_0^{-1} d^* \lambda. \tag{4.2$_b$}$$

Using the commutation $\Delta_0^{-1} d^* = d^* \Delta_1^{-1}$, we get

$$f = d^* \Delta_1^{-1} \lambda. \tag{4.2$_c$}$$

Using [10], we must lift the situation to the orthonormal frame bundle $O(G)$ of G. Set H the orthogonal group of the euclidean space $\mathcal{G}$, and set $\mathcal{H}$ its Lie algebra then $O(M) = H \times G$. The explicit expression of the Christoffel symbol characterizing the Levi–Civita connection has been computed in [3] formula $(2.1)_a$, as

$$\nabla_a X = \frac{1}{2} \operatorname{ad}(a)(X) := aX - Xa, \quad \text{then } \operatorname{ad}(a) \in \mathcal{H}, \tag{4.2$_d$}$$

and the lift $\tilde{a}_{i,j}$ of the vector field $a_{i,j}$ to $H \times G$ is defined as

$$\tilde{a}_{i,j}(\gamma, g) = \left(\frac{1}{2} \operatorname{ad}(a_{i,j})\gamma, a_{i,j} g \right).$$

The lifted Laplacian $\tilde{\Delta}_0$ of Δ_0 defines the process $(\gamma_{\tilde{b}}(t), g_b(t))$, where $\gamma_{\tilde{b}}(t)$ satisfies the system of equations

$$\begin{aligned} d\gamma_{\tilde{b}}(t) &= \frac{1}{2} \left(\sum_{i<j} \operatorname{ad}\big(a_{ij} \circ d\tilde{b}_{i,j}(t)\big) \right) \gamma_{\tilde{b}}(t), \qquad \gamma_{\tilde{b}}(0) = \text{Identity}, \\ db(t) &= \gamma_{\tilde{b}}(t)\big(d\tilde{b}(t)\big), \end{aligned} \tag{4.2$_e$}$$

and $g_b(t)$ is given in $(4.1)_b$ (see [1, Section 3]). According to [10, Proposition 2.4.1],

$$\Delta_1 = \tilde{\Delta}_0 - \text{Ricci}.$$

Given a differential form $\omega \in C^0(G; \mathcal{G})$, we have

$$\big(\exp(t\Delta_1)\omega\big)(g_0) = E\left(\exp\left(-\frac{d-1}{4} t \right) \times \gamma_{\tilde{b}}(t)\big(\omega\big(g_b(t) g_0\big)\big) \right), \tag{4.2$_f$}$$

and

$$\big(\Delta_1^{-1}\omega\big)(g_0) = \int_0^\infty \big(\exp(t\Delta_1)\omega\big)(g_0)\, dt.$$

In order to realize $(4.2)_c$ we have to apply d^* to $(4.2)_f$, that is to differentiate one-time relatively to the initial condition g_0; it is well known that such derivation can be obtained by a Girsanov transformation (see [11, p. 245]);

$$\big(d^* \exp(t\Delta_1)\omega\big)(g_0) = \sum_{i<j} E\left(\frac{b_{i,j}(t)}{t} \times \exp\left(-\frac{d-1}{4}t\right) \times \big(\gamma_{\tilde{b}}(t)\big(\omega(g_b(t)g_0)\big)\big)^{i,j}\right). \quad (4.2)_g$$

Set $\pi_t(g)$ the probability density of the law of $g_b(t)$ relatively to the measure σ. We deduce from $(4.2)_g$ that

$$\|K\|^p_{L^p_\sigma(G;\mathcal{G})} \leqslant \int_0^\infty \exp\left(-p\frac{d-1}{4}t\right) dt \int_G \left[\frac{\pi_t(g)}{\sqrt{t}}\right]^p \sigma(dg);$$

finally

$$\int_G \left[\frac{\pi_t(g)}{\sqrt{t}}\right]^p \sigma(dg) \simeq \frac{1}{t^{\frac{p}{2}}} \int_{R^{d'}} \exp\left(-p\frac{|\xi|^2}{2t}\right) \frac{d\xi}{t^{\frac{pd'}{2}}} \simeq t^{-\frac{p}{2}-(p-1)\frac{d'}{2}}. \qquad \square$$

Theorem. *Let φ be a C^1 function on the d-dimensional unit sphere S. Then*

$$\int_S \exp(\varphi)\, d\sigma \leqslant \int_S \exp\big(\|K\|_{L^1_\sigma(G;\mathcal{G})} \|\nabla\varphi\|_{\mathcal{G}}\big)\, d\sigma \times \exp\left(\int_S \varphi\, d\sigma\right). \quad (4.3)_a$$

Moreover, there exist two constants c_1, c_2 such that

$$\exp\big(\|\varphi\|_{L^\infty(S)}\big) < \left(c_1 + 2\int_S \exp\big(c_2\|\nabla\varphi\|_{\mathcal{G}}\big)\, d\sigma\right) \times \exp\left(\int_S \varphi\, d\sigma\right). \quad (4.3)_b$$

Proof. By substracting a constant to φ we reduce to the case where the mean value vanishes: then, expanding in Taylor serie the exponential, we get that

$$\int_S \exp(\varphi)\, d\sigma = \sum \frac{1}{p!} \int_S |\varphi|^p\, d\sigma.$$

By convexity of the norm L^p, we have

$$\|\nabla\varphi * K\|_{L^p} \leqslant \|\nabla\varphi\|_{L^p} \times \|K\|_{L^1}.$$

Then, using $(4.2)_a$ we obtain $(4.3)_a$.

Let us proceed to the proof of $(4.3)_b$; set $q > d'+1$ the conjugate exponent of $p < \frac{d'+2}{d'+1}$. Then, using $(4.2)_a$ together with Hölder's inequality,

$$\|\varphi\|_{L^\infty} \leqslant \|K\|_{L^p} \|\nabla\varphi\|_{L^q}. \quad (4.3)_c$$

On the other hand,

$$\exp\bigl(\|\varphi\|_{L^\infty}\bigr) = \sum_{n<q} \frac{1}{n!}\|\varphi\|_{L^\infty}^n + \sum_{n\geqslant q} \frac{1}{n!}\|\varphi\|_{L^\infty}^n.$$

Using $(4.3)_c$,

$$\sum_{n\geqslant q} \frac{1}{n!}\|\varphi\|_{L^\infty}^n \leqslant \sum_{n\geqslant q} \frac{1}{n!}\|K\|_{L^p}^n \|\nabla\varphi\|_{L^q}^n,$$

where

$$\|\nabla\varphi\|_{L^q}^n = \left[\int_S \|\nabla\varphi\|_{\mathcal{G}}^q \, d\sigma\right]^{\frac{n}{q}},$$

and, using the fact that $n \geqslant q$, it yields that

$$\left[\int_S \|\nabla\varphi\|_{\mathcal{G}}^q \, d\sigma\right]^{\frac{n}{q}} \leqslant \int_S \|\nabla\varphi\|^n \, d\sigma.$$

Hence,

$$\sum_{n\geqslant q} \frac{1}{n!}\|\varphi\|_{L^\infty}^n \leqslant \int_S \exp\bigl(\|K\|_{L^p} \times \|\nabla\varphi\|_{\mathcal{G}}\bigr)\, d\sigma.$$

In $(4.3)_b$, we choose,

$$c_2 := \|K\|_{L^p} > \|K\|_{L^1}.$$

Consider the polynomial

$$P(\xi) = \sum_{n<q} \frac{1}{n!}\xi^n, \quad \xi > 0.$$

Then,

$$\lim_{\xi\to+\infty} \exp(-\xi)P(\xi) = 0.$$

Set R such that

$$P(\xi) \leqslant \frac{1}{2}\exp(\xi), \quad \xi > R.$$

Then,

$$P(\xi) \leqslant \exp(\xi) - P(\xi), \quad \xi > R.$$

Finally, we take

$$c_1 := \max_{\xi \in [0,R]} P(\xi),$$

which concludes the proof of $(4.3)_b$. $\square$

4.1. Exceptional set of radius

Set

$$\varphi_r(\sigma) = f(r\sigma), \quad \text{and note that} \quad \|\nabla \varphi_r\|_{\mathcal{G}} = \frac{1}{r}\|\nabla f\|_{\mathcal{G}}.$$

Then, by $(2.3)_c$,

$$\int_0^\infty r^{d-1}\,dr \int_S \exp\bigl(cr^{-\gamma}\|\nabla\varphi_r\|^\gamma - \varphi_r\bigr)\,d\sigma \leqslant \int_{R^d} \exp\bigl(c\bigl\|\nabla f(x)\bigr\|^\gamma - f(x)\bigr)\,dx < +\infty.$$

Set

$$\Theta = \biggl\{r \in [1,+\infty[;\ \int_S \exp\bigl(cr^{-\gamma}\|\nabla\varphi_r\|^\gamma - \varphi_r\bigr)\,d\sigma < 1\biggr\}. \tag{4.4$_a$}$$

Then,

$$\int_{\Theta^c} r^{d-1}\,dr < \infty. \tag{4.4$_b$}$$

Using Cauchy–Schwarz inequality,

$$\biggl[\int_S \exp\biggl(\frac{c}{2}r^{-\gamma}\|\nabla\varphi_r\|^\gamma\biggr)\,d\sigma\biggr]^2 \leqslant \int_S \exp\bigl(cr^{-\gamma}\|\nabla\varphi_r\|^\gamma - \varphi_r\bigr)\,d\sigma \times \int_S \exp(\varphi_r)\,d\sigma,$$

and, in particular,

$$\biggl[\int_S \exp\biggl(\frac{c}{2}r^{-\gamma}\|\nabla\varphi_r\|^\gamma\biggr)\,d\sigma\biggr]^2 \leqslant \int_S \exp(\varphi_r)\,d\sigma, \quad \forall r \in \Theta. \tag{4.4$_c$}$$

Using $(4.3)_a$, setting $\tilde{c} := \|K\|_{L^1_\sigma(G;\mathcal{G})}$, we have

$$\biggl[\int_S \exp\biggl(\frac{c}{2}r^{-\gamma}\|\nabla\varphi_r\|^\gamma\biggr)\,d\sigma\biggr]^2 \leqslant \int_S \exp\bigl(\tilde{c}\|\nabla\varphi_r\|\bigr)\,d\sigma \times \exp\biggl(\int_S \varphi_r\,d\sigma\biggr), \quad \forall r \in \Theta.$$

Using $(3.2)_e$,

$$\left[\int_S \exp\left(\frac{c}{2}r^{-\gamma}\|\nabla\varphi_r\|^\gamma\right) d\sigma\right]^2 \leqslant \exp\left(c'_\gamma r^{\frac{\gamma}{1-\gamma}}\right) \times \int_S \exp\left(\tilde{c}\|\nabla\varphi_r\|\right) d\sigma, \quad \forall r \in \Theta.$$

Finally,

$$\int_S \exp\left(\frac{c}{2}r^{-\gamma}\|\nabla\varphi_r\|^\gamma\right) d\sigma \leqslant \sup(A, B) \quad \text{where } A := \exp\left(c'_\gamma r^{\frac{\gamma}{1-\gamma}}\right), \tag*{$(4.4)_d$}$$

and

$$B = \int_S \exp\left(c_2\|\nabla\varphi_r\|\right) d\sigma.$$

Theorem. *We have*

$$\|\varphi_r\|_{L^\infty(S)} \leqslant c''_\gamma r^{\frac{\gamma}{1-\gamma}}, \quad \forall r \in \Theta. \tag*{$(4.5)_a$}$$

Proof. Looking to the inequality $(4.4)_d$, we have two cases to consider, either $A \geqslant B$ or $A < B$.

In the first case we have

$$\exp\left(c'_\gamma r^{\frac{\gamma}{1-\gamma}}\right) \geqslant \int_S \exp\left(c_2\|\nabla\varphi_r\|\right) d\sigma;$$

we conclude by using $(4.3)_b$ and $(3.2)_e$.

In the second case, we have

$$\int_S \exp\left(\frac{c}{2}r^{-\gamma}\|\nabla\varphi_r\|^\gamma\right) d\sigma \leqslant \int_S \exp\left(c_2\|\nabla\varphi_r\|\right) d\sigma. \tag*{$(4.5)_b$}$$

Consider the set

$$\mathcal{E} := \left\{s \in S;\ \frac{c}{2}r^{-\gamma}\|\nabla\varphi_r\|^\gamma \geqslant 2c_2\|\nabla\varphi_r\|\right\} = \left\{s \in S;\ \|\nabla\varphi_r\| \geqslant c_3 r^{\frac{\gamma}{1-\gamma}}\right\}, \tag*{$(4.5)_c$}$$

and set

$$\zeta = \int_{\mathcal{E}} \exp\left(c_2\|\nabla\varphi_r\|\right) d\sigma,$$

then

$$\int_S \exp\left(c_2\|\nabla\varphi_r\|\right) d\sigma = \int_{\mathcal{E}} + \int_{\mathcal{E}^c} \leqslant \zeta + \exp\left(c_3 r^{\frac{\gamma}{1-\gamma}}\right) \tag*{$(4.5)_d$}$$

and

$$\int_{\mathcal{E}} \exp\left(\frac{c}{2} r^{-\gamma} \|\nabla\varphi_r\|^{\gamma}\right) d\sigma \geqslant \int_{\mathcal{E}} \exp\big(2c_2 \|\nabla\varphi_r\|\big)\, d\sigma. \tag*{$(4.5)_e$}$$

By Cauchy–Schwarz inequality,

$$\int_{\mathcal{E}} \exp\big(2c_2 \|\nabla\varphi_r\|\big)\, d\sigma \geqslant \left[\int_{\mathcal{E}} \exp\big(c_2 \|\nabla\varphi_r\|\big)\, d\sigma\right]^2 = \zeta^2.$$

Using $(4.5)_b$ together with $(4.5)_e$ and $(4.5)_d$, we get

$$\zeta^2 \leqslant \zeta + \exp\big(c_3 r^{\frac{\gamma}{1-\gamma}}\big),$$

which implies that

$$\zeta < \exp\left(\frac{1}{2} c_3 r^{\frac{\gamma}{1-\gamma}}\right).$$

Going back to $(4.5)_d$ we conclude $(4.5)_a$ by using $(4.3)_b$. □

5. Riesz transform on the unit ball

In the last section we have given the wanted estimate on all R^d at the exception of a very thin set of spheres corresponding to the spheres which have their radius in Θ; we fill these missing places by using now Riesz transform on balls of R^d.

Let $P(x, y)$ and $G(x, y)$ denote, respectively, the Poisson kernel and the Green function of the d-dimensional ball B of radius 1, that is,

$$P(x, y) = \frac{1 - \|x\|^2}{\omega_d \|x - y\|^d}, \quad y \in \partial B = S, \tag*{$(5.1)_a$}$$

and, for $d \geqslant 3$,

$$G(x, y) = \frac{1}{d(d-2)\omega_d}\left(\|x - y\|^{2-d} - \|y\| \times \left\|\frac{y}{\|y\|^2} - x\right\|^{2-d}\right), \tag*{$(5.1)_b$}$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$; remark that

$$G(x, y) = 0, \quad \forall x \in \partial B \quad \text{or} \quad \forall y \in \partial B \quad \text{and that } \Delta_x G(x, y) = \delta_y, \tag*{$(5.1)_c$}$$

where δ_y denotes the Dirac mass at y.

For $d = 2$ the Green function has the following expression:

$$G(x, y) = \frac{1}{2\pi}\left(\log\|x - y\| - \log\left\|\frac{1}{\|y\|} y - \|y\| x\right\|\right);$$

we shall not discuss later in detail the case $d = 2$, it is left to the reader.

Lemma. *We have*

$$\left\| \frac{y}{\|y\|^2} - x \right\| \geqslant \|y - x\|, \quad \forall x, y \in B. \tag{5.1$_d$}$$

Proof. Considering the plane containing x and y the statement is a statement in dimension 2; use the formalism of complex number set $y = \rho$ real and $x = \lambda = \xi + i\eta$ complex of modulus smaller than 1. We have to check that

$$(\lambda - \rho)(\bar{\lambda} - \rho) \leqslant \big(\lambda - \rho^{-1}\big)\big(\bar{\lambda} - \rho^{-1}\big) \quad \text{or} \quad -2\rho\xi + \rho^2 \leqslant -2\rho^{-1}\xi + \rho^{-2},$$

the inequality is satisfied for $\xi = 0$; it stays valid until the solution ξ_0 of the equation

$$-2\rho\xi_0 + \rho^2 = -2\rho^{-1}\xi_0 + \rho^{-2} \longrightarrow \xi_0 = \frac{\rho + \rho^{-1}}{2} > 1. \qquad \square$$

Lemma. *We have*

$$\forall x, y \in B, \quad \big\|\nabla_y G(x, y)\big\| \leqslant h(x - y) \quad \textit{where } h(z) := c\|z\|^{1-d} \times 1_{\|z\|<2}. \tag{5.1$_e$}$$

Proof. For the terms where do not appear the differential of $\|y\|^{-2}$ the estimate results from $(5.1)_c$ and $(5.1)_d$; the differential of $\|y\|^{-2}$ contributes as

$$\frac{1}{\|y\|} \times \left\| \frac{y}{\|y\|^2} - x \right\|^{1-d}.$$

As the singularity is in $y = 0$, we can assume that $\|y\| < \frac{1}{2}$. Then for all $x \in B$, we have

$$\frac{1}{\|y\|} \times \left\| \frac{y}{\|y\|^2} - x \right\|^{1-d} \leqslant \frac{2}{\|y\|} \times \left\| \frac{y}{\|y\|^2} \right\|^{1-d} = \frac{2}{\|y\|} \times \|y\|^{d-1} \to 0 \quad \text{when } y \to 0. \qquad \square$$

Remark. The indicator function $1_{\|z\|<2}$ is needed to realize that $\|h\|_{L^p} < \infty$ for all $p < \frac{d}{d-1}$.

Theorem *(Riesz transform). Let f be a C^1 positive function on the d-dimensional ball B of radius* 1. *Then, for all $x \in B$,*

$$f(x) = \int_{\partial B} f(y) P(x, y)\, \sigma(dy) - \int_B (\nabla_y G) \cdot \big(\nabla f(y)\big)\, dy. \tag{5.2$_a$}$$

Proof. By the Green representation formula, for any $x \in B$, and for any f of class C^2, we have

$$f(x) = \int_{\partial B} f(y), P(x, y)\, \sigma(dy) + \int_B G(x, y) \Delta f(y)\, dy.$$

Decomposing the laplacian in a sum of second derivatives and making an integration by parts, taking into account the vanishing of the Green function at the boundary, it yields

$$f(x) = \int_{\partial B} f(y) P(x, y)\, \sigma(dy) - \int_{B} (\nabla_y G) \cdot \big(\nabla f(y)\big)\, dy,$$

approximating a C^1 function by a sequence of C^2 functions and passing to the limit we prove $(5.2)_a$. □

Theorem. *There exist two constants c_1, c_2 such that for any C^1 function φ on the unit ball B, we have*

$$\exp\big(\|\varphi\|_{L^\infty(B)}\big) < \left(c_1 + 2\int_S \exp\big(c_2 \|\nabla\varphi\|_{R^d}\big)\, dx\right) \times \exp\big(\|\varphi\|_{L^\infty(\partial B)}\big). \tag{5.3$_a$}$$

Proof. We use the decomposition $(5.2)_a$; the norm $L^\infty(B)$ of the Poisson integral is equal to the norm in $L^\infty(\partial B)$ as for all $x \in B$,

$$\int_{\partial B} P(x, y)\, \sigma(dy) = 1.$$

It remains to evaluate the Green kernel integral; set

$$\psi(x) = \|\nabla\varphi\|_{R^d} \times 1_B(x).$$

Then, using $(5.1)_e$,

$$\left\| \int_B (\nabla_y G) \cdot \big(\nabla\varphi(y)\big)\, dy \right\|_{L^\infty(B)} \leqslant \|\psi * h\|_{L^\infty(R^d)}. \tag{5.3$_b$}$$

As $h \in L^p(R^d)$ for all $p < \frac{d}{d-1}$, we proceed as for the proof of $(4.3)_b$. □

Theorem. *Set*

$$\varphi_r(x) = f(rx), \quad x \in B,\ r \in \Theta. \tag{5.4$_a$}$$

Then,

$$\|\varphi_r\|_{L^\infty(B)} \leqslant c''_\gamma r^{\frac{\gamma}{1-\gamma}}. \tag{5.4$_b$}$$

Proof.
We proceed as for the proof of $(4.5)_a$. □

Proof of main theorem. Then $(2.3)_b$ results from $(5.4)_b$ combined with the fact that $(4.4)_b$ implies that

$$\Theta \cap [r, 2r] \quad \text{is always non-empty for } r > r_0.$$

That is, we conclude that for x sufficiently large

$$f(x) \leqslant \sup_{y \in B(0,\|x\|)} f(y) \leqslant c''_\gamma \|x\|^{\frac{\gamma}{1-\gamma}},$$

which proves $(2.2)_b$. □

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Appendix A

The one-dimensional version of theorem was proved in [13]. For the sake of completeness of this paper, we supply a sketch of the original proof.

Theorem. *Let f be a C^1 real positive function such that there exists $\gamma > 1$ and $c > 0$ such that*

$$\int_0^\infty \exp\bigl(c\bigl(f'(x)\bigr)^\gamma\bigr) \exp\bigl(-f(x)\bigr)\, dx < +\infty.$$

Then, there exists $\tilde{c} > 0$ such that for all $x > 0$ sufficiently large,

$$f(x) < \tilde{c} x^{\frac{\gamma}{\gamma-1}} f.$$

Proof. For all $n \in N$, define

$$\alpha_n = \inf\bigl\{x \geqslant 0:\ f(x) \geqslant n\bigr\}.$$

Write

$$\begin{aligned}\int_0^\infty \exp\bigl(c\bigl(f'(x)\bigr)^\gamma\bigr) \exp\bigl(-f(x)\bigr)\, dx &= \sum_{n=0}^\infty \int_{\alpha_n}^{\alpha_{n+1}} \exp\bigl(c\bigl(f'(x)\bigr)^\gamma\bigr) \exp\bigl(-f(x)\bigr)\, dx \\ &\geqslant \sum_{n=0}^\infty \exp\bigl(-(n+1)\bigr) \int_{\alpha_n}^{\alpha_{n+1}} \exp\bigl(c\bigl(f'(x)\bigr)^\gamma\bigr)\, dx.\end{aligned}$$

Consider the functional

$$J_n(f) := \int_{\alpha_n}^{\alpha_{n+1}} \exp\bigl(c\bigl(f'(x)\bigr)^\gamma\bigr)\, dx.$$

We want to find the minimum of $J_n(f)$ over all the C^1 positive decreasing functions $f : [\alpha_n, \alpha_{n+1}] \mapsto R$ such that

$$\int_{\alpha_n}^{\alpha_{n+1}} f'(x)\,dx = 1.$$

By the Lagrange multipliers method it suffices to find the minimum of the functional

$$J_n(f) - \lambda \int_{\alpha_n}^{\alpha_{n+1}} f'(x)\,dx,$$

where λ is a constant. The corresponding Euler–Lagrange equation is

$$\gamma c f'(x)^{\gamma-1} \exp\bigl(c\bigl(f'(x)\bigr)^\gamma\bigr) = \text{ const},$$

this implies that $f'(x)$ is a constant determined by the condition $\int_{\alpha_n}^{\alpha_{n+1}} f'(x)\,dx = 1$. If we denote $l_n = \alpha_{n+1} - \alpha_n$ we obtain that the minimum of $J_n(f)$ is reached when $f'(x) = l_n^{-1}$.

Thus,

$$\int_0^\infty \exp\bigl(c\bigl(f'(x)\bigr)^\gamma\bigr)\exp\bigl(-f(x)\bigr)\,dx \geqslant \sum_{n=0}^\infty l_n \exp\bigl(-(n+1) + c l_n^{-\gamma}\bigr).$$

By hypothesis, this serie is convergent. Hence, $l_n \exp(-(n+1) + c l_n^{-\gamma})$ converges to 0 as $n \to \infty$. This implies that there exists $n_0 > 0$ such that for all $n \geqslant n_0$,

$$(n+1) - c l_n^{-\gamma} > 0.$$

In particular, for any $q \geqslant n_0$, we have that

$$\sum_{n=n_0}^q l_n = \alpha_{q+1} - \alpha_{n_0} > \sum_{n=n_0}^q \frac{c^{1/\gamma}}{(n+1)^{1/\gamma}} > c^{1/\gamma} \int_{n_0+1}^{q+1} \frac{1}{x^{1/\gamma}}\,dx.$$

This implies that $\alpha_{q+1} > c_\gamma (q+1)^{\frac{\gamma-1}{\gamma}}$. But if x is such that $x < \alpha_{q+1}$, then $f(x) < q+1$. Hence, the desired result follows. $\square$

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